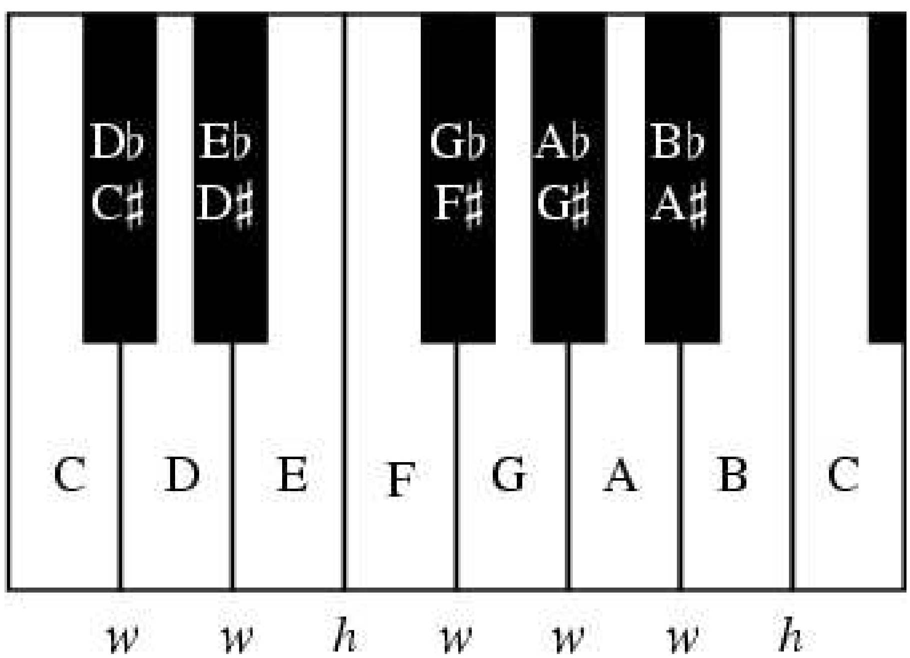
History of Mathematical Approaches to Western Music



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February 15, 2003

Abstract

Two simultaneous sounds are pleasing if the ratio of their frequencies is a rational number with small numerator and denominator. The octave interval denotes two sounds with ratio 1/2. It is impossible to perfectly subdivide 1/2 into a set of smaller intervals using only rational intervals. Various approximations (scales) are discussed. Also, an analytic theory for music composition is presented.

Contents

1 Introduction

Generally, we don't think of mathematics when we engage in music listening, and similarly we don't think about music when proving a theorem. Nevertheless, mathematics and music have been married by Pythagoras, and never got divorced. During the course of subsequent history, new music theories, ideas and notation have been invented by music theorists. Often, the goal of these systems was to characterise consonance and dissonance, and divide the interval of an octave into intervals that maximise consonance. The ultimate intention of these theories was to define rules according to which pleasurable music can be created. The rules of counterpoint developed during the Renaissance are one example of such system (sec. 5.1). In the twentieth century, more complex mathematical theories were invented. At the end of this historical overview, I introduce one of such theories: Composition with Pitch-Classes (sec. 6).

2 Psychology and philosophy

There are several qualitative aspects that are shared by mathematics and music[1]. In both disciplines there are no rules which specify *exactly* what to do in order to succeed (in proving a theorem, or in creating beautiful music). At most, there are only guidelines. These come in the form of new ideas or "hunches" that may be tried, as well as past experiences or "tricks" that have worked before and are worth a shot again.

As well, both processes require the brain to organise distinct ideas and experiences into one logical structure so that suddenly disconnected objects fall into place and "everything becomes obvious" [1]. For example, consider a piano player who learns a new piece one hand a time [1]. The music pattern may not be obvious during this process, since the music is designed to be played with both hands. Later, when the player had enough practice with separate hands and has decided to play both hands at the same time, the music pattern still may not be incidentally obvious. However, with additional practice, the notes fall into their correct places, so that the two melodies *match*, and suddenly music comes out! A similar process happens to a student of mathematics who studies many of its diverse fields separately (e.g. graph theory, statistics, linear algebra). Only at the end of the journey, he/she realizes that all of the seemingly unrelated disciplines are connected in intricate ways.

Similarly with mathematics and music: although at first glance they appear unrelated, their connection is evident upon closer inspection.

It is interesting to note that many famous mathematicians were also musicians or music theorists. In fact, treatises on music were written by Pythagoras, Ptolemy, Euclid, Boethius, Cardan, Huygens, Reimann, and Euler. For example, Euler published "Tentamen novae theoriae musicae" in 1739, in which he tried to make music [2]

> ... part of mathematics and deduce in an orderly manner, from correct principles, everything which can make a fitting together and mingling of tones pleasing.

3 Ancient Greeks

The fact that mathematics and music are related was clear to the Greeks. Music was included in the "quadrivium", subjects that are driven with logic, namely: number theory, geometry, astronomy, and music [3].

3.1 Pythagoras: The Father of Music Theory

The link between numbers and music was observed by Pythagoras (585-500 BC) by analysing the vibrations of strings of various lengths.

Imagine a taut string that is plucked such that it vibrates with frequency a (so as to produce sound). If we press with a finger at the midpoint of the string, so that the string would continue to vibrate in each of the produced halves, the frequency of the vibration of the string in each of the halves will *double* because the wavelength has decreased by a factor of two. In other words, the new frequency of the string vibration is $2a$. The frequency ratio between the new sound to the old sound is $2/1$[1]. Now imagine a similar experiment, but now we fix the string at two points, such that the string is divided into three equal parts. The original frequency a now triples for each of the segments. The frequency ratio between the notes $2a$ and $3a$ is $3/2$[2]. Pythagoras noticed that if the ratio between any two

[1] This interval is called octave or 'diapason'.

[2] This interval is called a fifth, or 'diapente'. A fourth is 4/3, also called 'diatessaron'.

note frequencies can be represented by a rational number p/q, where p and q are small integers, then the two notes are *consonant*. That is, if voiced one after the other they would create a pleasing change in sound[3].

Consider another scenario in which a woman is swinging a wire that connects a telephone handset and a telephone. When the woman starts swinging the wire slowly, the whole wire moves up and down. If she proceeds to swing increasingly faster, for a moment the wire will become "confused", but soon it would divide into two parts, with a stationary node in the middle (so that the whole waveform would look like one cycle of a sine wave). In this scenario she has not fixed the string in the middle by holding it, yet achieved the same frequency doubling result as in the previous example. If she continues increasing the speed of swinging, the string will divide into three, then four, *etc.*equal segments, each time the frequency increasing by an integer factor relative to the initial frequency a. These new frequencies are called *overtones* or *harmonics* of the base frequency a.

What Pythagoras did *not* know is that when a taut string vibrates, it vibrates at all of its overtones *at the same time!*[4]. However, the higher the overtone the smaller is the intensity of the vibration in that frequency. The sum of the vibrations of the overtones, that is, what we hear when the string is plucked, is called the 'timbre' of the musical instrument. It is therefore only possible to hear pure vibrations of a particular frequency (without overtones) with a use of a computer.

Thus, in practice, when we hear a sound, we automatically hear the sound intervals 1/1, 2/1, 3/1, 3/2. From here, it is now clear why melody 1/1, 3/2, 2/1, 3/1 sounds pleasing to the ear. It is almost as if we played only the note 1/1 four times.

3.2 The Tetraktys

Pythagoras and his followers have cherished the "tetraktys [quaternary] of the decad", the fourth triangular number $10 = 1 + 2 + 3 + 4$, which can be represented with the following triangle:

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   *
  * *
 * * *
* * * *
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This symbol stood for the four elements: fire, water, air and earth, and was the symbol upon which Pythagoreans swore their *oaths*[3].

In the tetraktys, the basic string length ratios are found, 2/1, 4/3, 3/2, *etc.*From [4] (p.12), a later Pythagorean Theon of Smyrna (2nd century A.D.) writes,

> The importance of the quaternary...is great in music because all of the consonances are found in it. But it is not only for this reason that all Pythagoreans hold it in highest esteem: it is also because it seems to outline the entire nature of the universe. It is for this reason that the formula of their oath was: "I swear by the one who has bestowed the tetraktys to the coming generations, source of eternal nature, into our souls." The one who bestowed it was Pythagoras, and it has been said that the tetraktys appears indeed have been discovered by him.

3.3 Consonance and Dissonance

Throughout the history of music, the ideas of consonance and dissonance have undergone dramatic changes. In the Ancient Greece, consonant intervals were considered to be fourths, fifths, octaves and combinations: fourth-octave, fifth-octave, octave-octave, and in addition, any octave compounds of the above mentioned intervals.

In his work "Harmonic Introduction" (first century AD), Cleonides, supporting the views of Aristoxenus, writes [4] p.13,

> Of intervals the differences are five, in that they differ from one another in magnitude, and in genus, and as the symphonic from diaphonic [dissonant], and as the composite from the incomposite, and as the rational from the irrational...The symphonic intervals are the diatessaron, diapente, diapason, and the like [presumably the octave-compounds of these]. The diaphonic intervals are all those smaller than the diatessaron and those lying between the symphonic intervals...

[3] Greeks didn't consider voicing the notes simultaneously. History had it that only a thousand years later did the birth of polyphony (playing of simultaneous sounds) occur.

[4] But Rameau realized this fifteen centuries later.

3.4 The Diatonic Scale

In loose terms, the diatonic scale is a division of an octave into seven intervals. The modern diatonic scale now corresponds to the white keys of the piano (Figure 3.4) which uses an equal tempered tuning, see section 5.4. Up until the sixteenth century the tunings for the diatonic scale were in just intonation, a system where each interval is represented by a rational number.

The notes in the diatonic scale are denoted by letters A,B,C,D,E,F,G. Symbols # and b, called sharp and flat, respectively, modify the pitch of a note. A sharped note is raised half a step, and a flattened note is lowered a half-step.

In general, the precise frequencies of these values, and exactly by how greatly sharps and flats change them, remain undefined. These values are given a particular meaning upon the choice of scale or *tuning*.

3.5 Ancient Pythagorean Tuning

Observing the connection between numbers and sounds, Pythagoras went ahead to create a tuning for the diatonic scale based only on combinations of fifths.

The ancient Pythagorean tuning is the descending scale with notes [5]: $1/1, 8/9, 27/32, 3/4, 2/3, 16/27, 9/16, 1/2$[6][6]. Note that all ratios in a descending scale are less than 1. Also, notice that all intervals involved in this tuning can be created by combining (multiplying) intervals 8/9 (called 'second' or 'tone') and 243/256 (called 'hemitone').

Most likely, Pythagoras has produced this scale in the following manner. He started with a root note, and produced ascending and descending fifths, while transposing them in order to keep them in the same octave (i.e multiplying them by 2/1 (transpose up) or dividing them by 2/1 (transpose down), so that the value of the ratio is always at least 1 and at most 2).

Original Fifth	Transposed Fifth
27/8	27/8
9/4	9/16
3/2	3/4
1/1	1/1
2/3	2/3
4/9	8/9
8/27	16/27

A reorganisation the notes in the right column produces the Pythagorean scale.

Finally, notice that two hemitones don't produce a second,

$$(\frac{256}{243})^2 \approx 1.110 < \frac{9}{8}$$

.

The intervals descending between consecutive notes in the scale are: tone, hemitone, tone, tone, tone, hemitone, tone. Compare this scale to the Medieval Pythagorean tuning in Section 4.3.

The Pythagorean tuning, in a slightly modified form, has lasted for two millennia, until it was replaced by the Equal Tempered Tuning as well as the just intonation tuning developed by Ptolemy (but forgotten and rediscovered only during the Renaissance!).

3.6 The Greek Modes

In Ancient Greece the word "mode" stood for a scale [7]. Each mode was a rotation of the intervals $\{9/8, 9/8, 256/243, 9/8, 9/8, 9/8, 256/243\}$. Since there are seven notes here, each mode could effectively be the definition of the diatonic scale.

The lydian mode is one particular tuning of the diatonic scale which corresponds to the white keys on the piano, starting from F and ending on F (roughly; see section 5.4).

3.7 The Tetrachord

The tetrachord, which literally means "four strings" was the basic scale unit of Ancient Greece. The first and fourth strings were always tuned a fourth apart. The two strings in the middle depended on the "genus" and mode of the music. There were three kinds of genera[7]: the diatonic, chromatic, and enharmonic.

In the diatonic genus, the two middle intervals were two tones, and a semitone. The chromatic genus comprised a minor third (three semitones), and two semitones. In the enharmonic mode – (major third) two tones, and two quarter tones.

However, in practice these tunings probably were not exact [8], since

> Prior to Pythagoras there appears to be little evidence of a theoretical basis for the tuning of musical scales. Pythagoras

[5] In Ancient Greece all scales were descending.

[6] This notation means, for example, that the second note frequency is 9/8 times smaller than the base (root) note

[7] Note that "genera" is the plural of "genus"

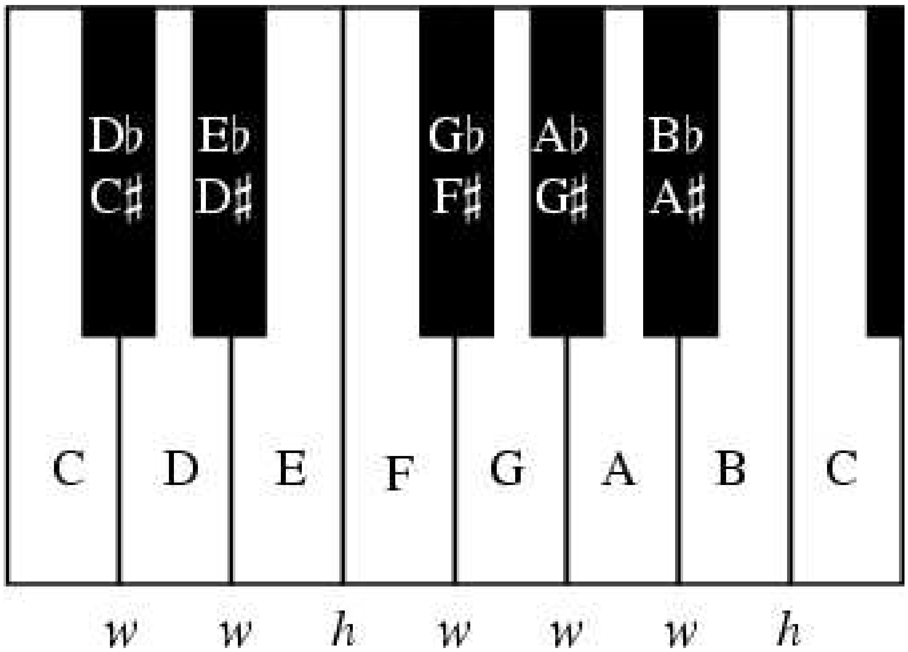

Figure 1: Note that in just intonation tunings the sharp and a flat is not necessarily the same (e.g $D\#$ is not the same as Eb). The modern piano is tuned according to the equal tempered tuning (section 5.4). This diagram is taken from [5].

> was involved with the science of harmonics which was separate from the practical art of music. In the absence of a theoretical basis for the tuning of scales the actual tuning can only have been empirical and probably varied widely.

3.8 Ptolemy's Tuning of the Diatonic Scale

The modern diatonic scale tuning using just intonation was developed by Ptolemy[9] in the second century AD. This scale is formed by three major triads, note triples with frequency ratios $4:5:6$. The major triads are $C, E, G;$, $F, A, C;$, G, B, D. All the intervals in this scale are a combination (product) of the intervals $9/8 = 1.125$, $10/9 \approx 1.111$ and $16/15$. The first two are called whole-steps, and the last—a half-step, or a semitone. Note that two half-steps are larger than one whole-step, $(16/15)^2 \approx 1.138$.

An alternative way to describe this scale is with the use of tetrachords. In this interpretation, the octave $(2/1)$ is divided into a tetrachord, followed by a second $(9/8)$ (aka whole-tone), followed again by a tetrachord. The total interval of the tetrachord is a fourth $(4/3)$, and a fourth combined with a fifth make up an octave $((4/3) \times (3/2) = 2/1)$. This is summarized visually as,

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octave = fourth | second | fourth
octave = fourth |      fifth
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Therefore, the definition of a second must be the difference between the fourth and the fifth! That is, $(3/2)/(4/3) = 9/8$.

The Ptolemaic tuning was ignored during the Medieval Period, and resurfaced again during the Renaissance [6]

> The diatonic scale is of ancient origin, but the particular tuning incorporated into modern just intonation (see below) is due to Ptolemy... It was rediscovered by Gaffurio in the late 15th century, from whom Zarlino learned about it, and it has remained the basic scale used in western music ever since (Jeans 1938, p.164).

3.9 Archytas: Dividing the Consonant Intervals Further

Another Pythagorean philosopher, Archytas of Tarentum (428-347 BC), who is most known for constructing the first flying machine[10][8], observed in his work "Proportions: Arithmetical, geometrical, harmonic" that it is not possible to divide the fifth (3/2), fourth (4/3), the octave (2/1), and the second (9/8), or in general an interval of the form $(n+1)/n$, into two equal intervals using rational numbers. That is, there exists *no* solution to,

$$(\frac{a}{b})^k = \frac{n+1}{n}$$

for $k > 1$, a, b, k, n integers.

Archytas also observed that the product of a harmonic mean of two elements with the arithmetic mean of the same two elements is equal to the square of the geometric mean of the same two elements. That is, for $x < y$, the harmonic mean h satisfies[9]

$$\frac{h-x}{x} = \frac{y-h}{y}$$

or equivalently,

$$h = \frac{2xy}{x+y}$$

the arithmetic mean m,

$$m = \frac{x+y}{2}$$

and the geometric mean g,

$$g = \sqrt{xy}$$

So that $g^2 = hm$. It can be shown that $a > g > h$.

Thus, by using this property, Archytas was able to divide the fifth 3/2 into the product of 5/4 (major third) and 6/5 (minor third), so that the fifth is divided into $6:5:4$. Similarly, the fourth 4/3 can be divided into the product of 7/6 and 8/7 so that the fourth becomes divided into $8:7:6$. The interval 7/6 can be thought of as a shrunken minor third, and the interval 8/7 as an enlarged whole tone [11].

The geometric beauty of this system lies in the fact that Archytas used different kinds of averages to create an almost "average" or "equal" division of the intervals.

3.10 Problems with Just Intonation

Assigning note frequencies according to rational numbers introduced some problems. For example, if we play twelve fifths in a row (twelve notes, between each adjacent pair the interval is a fifth, 3/2) we would get an interval $(3/2)^{12}$. If we take seven octaves, we get $(2/1)^7$. The interval between the two resulting notes is[10],

$$\frac{(3/2)^{12}}{(2/1)^7} = \frac{3^{12}}{2^{19}} = \frac{531441}{524288} = 1.013643265$$

i.e. the two notes are almost the same, but *not* exactly, and the two notes will not sound pleasurable. This dissonance (sound interval that hurts the ear) is called "comma of Pythagoras".

Curiously, as it was observed by the most eminent of the Pythagorean school, Philolaus of Tarentum (or Croton), who lived in fifth century AD [12] [11], two heminotes 256/243 and a comma of Pythagoras make up the whole tone 9/8,

$$(\frac{256}{243})^2 \times \frac{531441}{524288} = \frac{65536}{59049}) \times \frac{531441}{524288} = \frac{9}{8}$$

Listed below are problems similar to the comma of Pythagoras. Here a major third is 5/4, and a seventeenth is 5/1 (a seventeenth is equal to two octaves and a major third).

Comma of Didymus: (four fifths) vs. seventeenth.

Diesis: (3 major thirds) vs. octave

Shisma: (8 fifths and major 3rd) vs. 5 octaves.

In the 16th century, music theorists have attempted to fix the comma of Didymus and comma of Pythagoras, by changing the definition of a fifth, with the invention of the Mean Tone Scale (described in section 5.2).

Another problem with just intonation is that it is not possible to *transpose* the music to a different starting note. Such transformations can only be possible if the intervals between adjacent notes are all equal. Equal interval tunings are discussed in section 5.4.

[8] Archytas' machine, called "pigeon", invented in 425 BC, was powered by a jet propulsion system, and in one experiment flew 200 meters. However, once it landed, it could not take off again [10].

[9] The general harmonic mean for elements $x_1, x_2, \ldots, x_n$ satisfies $\frac{1}{h} = \frac{1}{n}(\sum \frac{1}{x_i})$.

[10] Every interval is also a note; simply, we assume that the root note is the lower point, and the upper point is the note of interest, such that the interval between the two notes is equal to the interval in question.

3.11 Performed Music in Ancient Greece

Contrary to the wealth of theoretical foundations that Ancient Greece has contributed to music, almost no record of the performed music has been preserved [13] p.4. It is known, however, that most of the music was improvised. Also, several instruments did not play simultaneously.

Surprisingly, Greek artists did not consider the value of music on its own merit; music was played always in combination with poetry reading. Since their poetry often conformed to a beat structure, the accompanying music was played to that natural beat.

4 Medieval Theory and Practice

The knowledge of ancient Greek music theory became available to medieval music theorists through the treatise "De Institutione Musica" written by Boethius (480-524 AD).

In medieval times, the west became Christian. Consequently, this has affected the music practice. For the next fifteen centuries the musical scene was divided into two categories: sacred chant (religious musical prayer) and secular (non-sacred, social, popular).

4.1 Sacred Chant

There were two categories for Christian service, the Divine Office and the Mass. The first was inherited from the Jewish Tradition, in which a community prays collectively, daily, every few hours. This tradition involves the singing of psalms (from Old Testament Book of Psalms), hymns and praise songs. The other service, called the Mass, is the most important Christian service, which symbolizes the Last Supper. In it the bread and wine is blessed and offered.

The chants are recorded with a specific form of notation. Unlike in the standard music notation, where five lines are used for the staff, only four are used for chant recording. Also, instead of oval-shaped symbols for the notes, square symbols are used. In general, the duration of the notes (called 'neumes' in this context) is the same. The chant melodies usually remain within one octave, and the chant text is written under the neumes. When every syllable corresponds to a its own neume, the section is called syllabic. When several neumes are sung to the same syllable, the section is called "melisma".

As time progressed, modifications were made to the existing and documented chants by the process of troping, and sequencing. The three kind of troping processed were: text added to existing melismas, music extending melismas or adding new ones, new words and music added to a regular chant. Sequencing on the other hand refers to writing new words under long melismas, to for either artistic reasons, or to aid memorizations. Although troping was supported by the Church, most sequences were banned.

4.2 Secular Songs

Bordering the fine line between religious and secular was the "conductus", a style of song on sacred and non-sacred subjects, with metrical verses written in Latin, sung with a rhythmic pulse (analogous to a march). Another type of secular song was known as the "chanson de geste", an epic narrative poem about national heroes, sung to a simple melody. The people who sang chansons de geste were called "jongleurs" or "ministrels", a class of professional musicians that appeared in the tenth century. Finally, "troubadours" were poet-composers living in the South of France, who, influenced by Arabic love poetry, created beautiful love ballads.

4.3 Medieval Pythagorean Tuning

By using the Pythagoras' method of combining fifths, medieval theorists developed a tuning based on ascending fifths *only* (recall that the Pythagoras used both ascending and descending fifths, section 6). The produced fifths were transposed down as many octaves as needed in order to keep the range within one octave. The resulting scale is $1/1, 9/8, 81/64, 4/3, 3/2, 27/16, 243/128, 2/1$[6].

Again, note that all intervals involved in this tuning can be created by combining (multiplying) intervals $9/8 = 1.125$ (second) and $256/243$ (hemitone).

The intervals between successive note pairs are as follows: tone, tone, tone, hemitone, tone, tone, hemitone. Compare this to the Ancient Pythagorean Tuning described earlier in Section 6.

Note that this sequence of intervals corresponds to the intervals between the notes F, G, A, B, C, D, E, F, as shown on the the diagram of the piano keyboard in Figure 3.4.

4.4 Guido of Arezza

Guido of Arezza (990-1050) significantly contributed to the music theory of the Medieval Ages. Guido used hexachords (chords containing six notes) to create a scale called "Gam"[11]

4.5 Note Names

Interestingly enough, before Guido, the notes did not have names, so the only way to practice singing a note was to sing it with some poetry. Guido had proposed the names "ut,re,mi,fa,sol,la" to name the notes in his hexachords (cords of 6 notes each, with intervals tone,tone,semitone, tone,tone[12]). Guido took these names from a hymn called "Ut queant laxis", with these words:

Ut que-ant la-xis re-sona-re fi-bris
Mi -ra ge-sto-rum
fa-mu-li tu-o-rum
Sol-ve po-lu-ti
La-bi-i re-a-tum
San-cte jo-an-nes

Note that the song phrases begin with the sounds assigned to the corresponding notes to which they were sung in this hymn. These names are still in use today, with 'ut' changed into 'do', and 'ti' added for the note B, taken from the word 'po-lu-ti' above.

4.6 Guido's Hexachord

In the Medieval Pythagorean tuning as described above (section 4.3), the hexachord of Guido can be taken to be the following chords: $\{G, A, B, C, D, E\}$, $\{C, D, E, F, G, A\}$, $\{F, G, A, Bb, C, D\}$ (here Bb is defined by decreasing B by a hemitone). The hexachord beginning on G was called hard or "durum", due to the natural B (that is, not flat), and the hexachord beginning on G is soft or "molle". The hexachord starting on "C" was called natural.

The "Gam" scale was based on seven hexachords which overlapped to produce a range of two octaves and a major 6th. Guido's Gam scale survived until 1500s, when it was replaced by a system of major and minor thirds (developed due to Ptolemy, and then rediscovered; see sec. 3.8).

4.7 Gregorian Modes

The Gregorian modes are named after Pope Gregory II (reigned 715-31)[13] p.16, and use some of the names of the ancient Greek modes. However, they are not the same: they are rotations of a diatonic scale C, D, E, F, G, A, B. They are called Dorian, Phrygian, Lydian, Mixolydian, *Aeolian, Locrian, and Ionian* which start on notes D, E, F, G, A, B, C respectively (note that Ionian is another name for the original diatonic scale). The last three names have been defined and added later by Henricus Glareanus, in his book "Dodecachordon" (1547).

Glareanus has also added a modified Dorian and Lydian modes, in which a B is replaced with Bb. These modifications are important, since for example, the modified Lydian mode is equivalent to the diatonic major tuning describe later in Section 4.8. Interestingly enough these modifications have been ignored for many centuries.

The first note of the above modes, about which the rotation is taken is called 'final'. These modes were called "authentic". A variation of these modes, with the same root name, but with a prefix 'Hypo' (e.g. Hypodorian) are the rotations such that the final note is in the middle, that is, a fourth note of the scale (which also happens to make a fourth with the first note).

4.8 Diatonic Major Tuning

This important tuning can be achieved from the Medieval Pythagorean Tuning (sec. 4.3 by transposing all the notes by a fifth down, (i.e. multiplying by 2/3). As for the notes F, G, A, B (they are $1/1, 9/8, 81/64, 4/3$ when F is the root), after transposing them down a fifth, transpose them up an octave (so as to remain within one octave's range).

If we order the notes of the scale F, G, A, B, C, D, E with intervals $1/1, 9/8, 81/64, 4/3, 3/2, 27/16, 243/128, 2/1$ as C, D, E, F, G, A, B with intervals $3/2, 27/16, 243/128, 2/1$, $1/1, 9/8, 81/64, 4/3, 3/2$ and then perform the above transformations, we will get $9/8, 9/8, 256/243$, $9/8, 9/8, 9/8, 256/243$ which is the major diatonic scale.

Note that this new scale which starts from C is in fact a "rotation" of the intervals in the F scale. This

[11] From this name came the saying "running the gamut". Literally it means to play *all* notes of the scale.

[12] In equal temperament, introduced lower, tone is the interval $2^{1/12}$. In just intonation, tone is 9/8 and semitone is whatever decided by the tuning, but approximately half of 9/8

is also suggested by the distribution of the piano keys (fig. 3.4).

4.9 Polyphony

In the ninth century, the first experiments with polyphony, *simultaneous* sounding of notes, began in double voice singing, called organum. In previous cases, any singing in unison has occurred for notes being either exactly the same frequency, or octaves apart. In the ninth century organum, new ideas in chant singing, for simultaneously voicing fourths and fifths, were born.

Polyphony began in the Church liturgy, as a modified process of troping (described earlier in section 4.1). Now, instead of writing new text under the neumes (notes), a new voice was written. The new voice consisted of notes of the original voice transposed either a fourth or a fifth down, an octave, or unchanged. This new polyphony is called the "Early Organum".

The polyphonic evolution of the plainchant (sacred chant) has reached its highest at the Notre Dame Cathedral which was build at the soaring Gothic style of the twelveth century.

In addition to polyphonic singing, The Notre Dame Organum produced a breakthrough in *rhythmic* notation. Recall that since rhythm was usually defined by the text of a chant, it was implicit. Now, explicit rhythmic styles were developed.

These new rhythmic styles were developed by famous composers Leonin (1135-1201) and Perotin[13]. The rhythms developed by them were six modes, or templates. For example, the first mode is "long, short"; the second is "short, long", the third one is "long-short, short, long", and so on. Here the time of "short" is half the time of "long".

Perotin and Leonin have also developed the genre called "Motet", which became the standard for both sacred and polyphonic music. The voice structure of the motet can be related to the Gothic structure of the buildings that were built during that period, particularly the Notre Dame Cathedral, where the motet was born[13]. In the Gothic style, the lowest arches contain smaller arches on their maxima, and the smaller arches themselves contain smaller arches. This is done in a symmetric fashion, so that the second level of arches go in pairs (every bottom arch holds a pair of arches), and third level arches go in triples (every second level arch-pair holds a triple-arch). In a similar way, a motet is structured. Usually sung in three voices, the motet has lower voices supporting faster middle voices, which in turn support ever faster higher voices.

4.10 Consonance and Dissonance

In the Medieval Times, the definition of consonance and dissonance was further developed. Boethius have categorized the relationship between intervals into three categories: "equisonae", "consonae", and "unisonae". By unisonae he meant two notes played at the same frequency (i.e. exactly the same notes), by equisonae or identical – notes that are octaves apart, and finally by consonae – diapente (fifth) and diatessaron (fourth). Boethius did not state if the notes are to be played together or one at a time.

In the tenth century, Flemish monk, composer and writer Hucbald (840-930) has redefined these terms. In his system "aequisonae" are notes that are either the same, or octaves apart, and "consonae" are all possible consonances (diatessaron, diapente, diapason (octave), diatessaron-diapason, diapente-diapason). Also, Hucbald makes a distinction between simultaneously played notes and separately played notes. In his terminology, consonance refers to simultaneously played notes, while "intervallum" or "spatium" refers to a melodic interval [4] p.17.

Subsequent modifications were made by John of Galrand in his treatise "De musica" (1100 AD). John categorized the consonances and dissonances into three types in each case. For the consonances, the three types of concords (consonant chords) are perfect, in-between, and imperfect. Similarly, for the discords: there are perfect discords, in-between discords and imperfect discords.

The perfect concords are octave and unison (the two voices can not be distinguished by ear), the in-between are fourth and fifth, and the imperfect are third (major: 5/4 and minor: 6/5) and the sixth (major: 5/3 and minor: 8/5) (the voices are clearly distinguishable by ear in these intervals).

Conversely, the perfect discords are semitone, triton, and ditone with diapente (major seventh), the imperfect – the ditone and diapente, and semiditone with diapente. Finally, the in-between intervals are the tone and semitone with diapente.

[13] Exact birth and death dates are unknown, but he has most likely was born in the middle of twelveth century, and dies in the beginning of the thirteenth century [14].

5 Renaissance

The development of the Motet in the twelfth century helped to carry the polyphonic ideas to the Renaissance of the fifteenth century. With polyphony becoming established, the fourteenth century emphasis of music shifted from melody to harmony and counterpoint (tension). With this change came an increasing use of intervals of imperfect consonances, namely, thirds and sixths (equivalently, major and minor thirds).

5.1 The Rules of Counterpoint

In the fourteenth century, a new system of consonance classification has emerged. The six categories introduced by John of Garland were now reduced to three: "perfect consonances", "imperfect consonances" and "dissonances". Now the fifth has been elevated from imperfect consonance to perfect, and major and minor sixth became imperfect consonances (which roughly corresponds to the in-between category of John). All other intervals are dissonances.

A special case is the fourth. Surprisingly, while being ranked as the first of the perfect consonances in the past, it is now treated as a special kind of dissonance or consonance. This is indeed very puzzling!

In his book "Writing on Music", Part II (1574), the famous mathematician Girolamo Cardano (1501-1576) (aka. Jerome Cardan) assigns the fourth to a special category of "median" intervals, ranked after "pluperfect","perfect","imperfect" intervals–but before the dissonances–in order of decreasing consonance[4] p.46. His "median" category includes intervals that are dissonant in themselves but consonant in their combination.

This new bias towards the fourth apparently was not supported theoretically, but instead was a result of several centuries of trial and error [4]p.46,47. In [4] p.48, James Tenney suggests that the nature of the dissonance of the fourth may lie in the analysis of its harmonics (the overtones).

Independently of reasons for not accepting the fourth into consonance, the newly established classification system of intervals gave the terminology to state a set of rules for composing pleasing music, known as "counterpoint" [4] p.40,46. The goal of these rules was to guide composers to correctly balance tension with release. The rules are as follows:

1 A piece should begin and end with a perfect consonance.

2 According to Cardan [4] p.46,

> ...when ambiguous intervals are used in the lower voices or in a two-voice composition, they dissonate in the same way as in the first rule by upsetting the composition's relationship, for they become dissonant sounds.

3 Consecutive parallel perfect consonances of the *same kind* are to be avoided.

4 However, consecutive *imperfect* consonances can be used freely.

5 Stepwise and contrary motion is preferred.

In his book "The Art of Counterpoint" (1588) [4]p.42, Zarlino commented on the importance of dissonant intervals that

> ...intervals that are dissonant produce a sound that is disagreeable to the ear and render a composition harsh and without any sweetness. Yet it is impossible to move from one consonance to another...without the means and aid of these intervals.

Again, he adds later that

> ...every composition, counterpoint, or harmony is composed primarily of consonances. Nevertheless, for the greater beauty and charm dissonances are used, incidentally and secondarily. Although these dissonances are not pleasing in isolation, when they are properly placed according to the precepts to be given, the ear not only endures them but derives great pleasure and delight from them...

5.2 The Mean Tone Scale

In the sixteenth century, with the new ideas of consonance and dissonance, Pythagorean tunings also were questioned. New tuning system systems started to emerge.

One of the new tunings is the Mean Tone Scale that was invented to eliminate both the comma of Didymus, as well as the comma of Pythagoras. In the comma of Didymus, four fifths vs. seventeenth, we have $(3/2)^4/5 = 81/80$. By changing the definition of fifth from $(3/2) = 1.5$ to $5^{1/4} = 1.495348781$, we remove the problem, so that the ratio in question becomes 1/1. The new type of "fifth" is called a mean fifth.

However, the Mean Tone Scale scale introduces a new type of dissonance, called a "wolf fifth". See [15].

5.3 Other Just Tunings

Extensions were made to the diatonic tuning to include the rest of the flats and sharps so as to produce the *Chromatic Scale* consisting of 12 notes. Also, Franchino Gaffurio has uncovered the just tuning of the diatonic scale invented earlier by Ptolemy (see 3.8), but being too conservative opposed it [16]. It was Ludovico Fogliano who published it in "Musica Theorica" (1529) and Zarlino who published it in "Intutioni Armoniche" (1558).

Since the Ptolemaic tuning used major and minor thirds[14] in its triads (notes in ratio $4 : 5 : 6$), this scale was better suited for the new ideas of consonance and dissonance of the Renaissance, as compared to the Pythagorean tuning.

Also, like Archytas of Ancient Greece (sec. 3.9), Zarlino has observed the relationship between the harmonic and arithmetic means and interval subdivisions. P. A. Fraser comments [16],

> Zarlino observed that the arithmetic mean 3 between 2 and 4 divides an octave into a fifth and a fourth, $2 : 3 : 4$. (Or $6 : 9 : 12$.) Alternatively, the harmonic mean 8 between 6 and 12 divides the octave into a fourth and a fifth, $6 : 8 : 12$. Similarly, the arithmetic mean 5 between 4 and 6 divides a fifth into major and minor thirds, $4 : 5 : 6$, whereas the harmonic mean 12 between 10 and 15 divides the fifth into minor and major thirds, $10 : 12 : 15$. Furthermore, the arithmetic mean of a major third, $4 : 5$ or $8 : 10$, divides it into major and minor tones, $8 : 9 : 10$. Zarlino saw this result as 'truly miraculous'.

5.4 The Equal Tempered Scale

To alleviate the problems connected with just intonation tunings (sec. 10), a new kind of tuning based on exponents has emerged. This new tuning is called the Equal Tempered Scale.

Suppose we want to divide the octave 2/1 into M equal intervals a. Then, combining a with itself M times, must give us 2/1, namely, $a^M = 2$. So, $a = 2^{1/M}$. We want as many notes in our scale to fall as close as possible to notes with frequencies equal to ratios with small integer numerators and denominators. Figure 15 shows that choices $M = 12$, $M = 19$ and $M = 31$ are reasonable. Western music adopts $M = 12$.

The first to investigate the idea of a tuning based on a subdivision into equal intervals was Vincenzo Galilei (1525-1591), the father of Galileo Galilei[11][15]. However, because he tried to use rational numbers (in particular the interval 18/17), it did not work (see section 8 for the impossibility of subdivision of $(n+1)/n$ into equal rational intervals). However, the system based on semitones equal to interval 18/17 is still in use today for the tuning of the lute, the viola, and similar kind of instruments[18]. In this system, the octave is twelve semitones, the fifth is seven semitones, and the fourth is five semitones.

It was the Dutch mathematician and engineer Simon Stevin (1548-1620) who came up with the idea of using uniform steps of size $2^{1/12}$. Jay Murray Barbour, in his book Tuning and Temperament (Michigan State College Press, 1953), remarks about Stevin's achievements in music[18],

> In his days only a mathematician (and perhaps only a mathematician not fully cognizant of contemporary musical practice) could have made such a statement...It is refreshingly modern, agreeing completely with the views of advanced theorists and composers of our day.

Fokker adds in [18],

> He [Stevin] had no bump for the plain simplicity of small integer numbers. In his treatise on arithmetic (Work V) he had

[14] Note that a minor third is equivalent to major sixth, but with the order of the notes reversed.

[15] Vincenzo Galilei studied music theory in Venice under Zarlino, with whom he later had a dispute about music theory. Before, it was though that in the same way that the ratio of lengths 2/1 of two vibrating strings yields the sound interval 2/1, so would be the case for the the ratio of *strings tensions* for two strings of equal length, tuned an octave apart. Vincenzo has conducted an experiment using hanging weights to show that the tension ration is not 2/1 as thought before, but 4/1 [17].

> explained that there are "no absurd, irrational, irregular, inexplicable, or surd numbers" (see this edition, Vol. II B, p. 532, also Vol. I, p. 23). For him a number like $2^{7/12}$ is as good as any other, say 3/2. If anybody should doubt that the sweet consonance of the fifth could be compatible with so complicated a number, then, says Stevin, rather haughtily and aggressively, he is not going to take pains to correct the inexplicable irrationality and absurdity of such a misapprehension. He repudiates the Pythagorean values for the intervals (3/2 for the fifth, 9/8 for the second, 81/64 for the major third, 4/3 for the fourth) on the ground that they lead up to the ratio 256/243 for a semitone (the minor limma). This, when subtracted from a whole tone (9/8), leaves another semitone with a ratio very close to 256/240. Stevin remarks that this major semitone is all but a quarter larger than the previous minor semitone (the differences of 243 and 240 from 256 being 13 and 16 respectively). All semitones having to be equal, the initial assumption of 3/2 for the ratio of the fifth must be wrong.

With the idea of dividing the octave into intervals of the form $2^{1/M}$ established, another question was raised of what should be the value of M. The famous physicist and mathematician Christian Huygens (1629-1695) in his letter "Lettre touchant le cycle harmonique" (1691) [19] and in "Novas cyclus harmonicus" (1724), together with Nicola Vicentino (1511-1576)[20], has advocated the use of *31-tone system*.

The choice of $M = 12$ has not been absolute in non western cultures. Persians divided their octave into 24 unequal intervals; Arabs into 16, but mostly used octave, fifth and *quarter* notes; Indians divided the scale into 22 notes, but only used 7 intervals; Chinese – into 12 equal intervals, but used mostly pentatonic scale (five particularly selected notes) [21]. In the modern times, the topic of dividing the octave into more than 12 notes, termed *microtuning*, is an active subject of research.

6 Twentieth Century Formalized Music Theory

Musical ideas that were developed through history can be formalized in mathematical language so as to aid in harmonic and melodic composition. The key idea in the theory I am about to present is that given a collection of notes that are voiced simultaneously (a chord), we may count which intervals occur in it by looking at the intervals between each pair of notes that make this chord. If two different chords involve the same intervals, they should sound similar. Thus, the passage from one to the other should be a pleasing transition to the ear.

Thus, given a chord, we could perform transformations on it without loosing its interval structure, and we could create a sequence of similar-sounding yet different chords.

This process can be generalized to operate not just on a single set of notes, but on a two dimensional array of notes. The rows of the array can be thought of as voices. Hence, as we sweep the array horizontally (left to right), we play the notes in the array cells *melodically*. As we slice the array vertically, we have the notes that are executed simultaneously by all voices, i.e. *harmonicaly*. The operations on such arrays, if they are to preserve array's musical content, must not alter the total content in the columns and rows.

I will now delve into detail. We will be working with the equal temperament scale with 12 notes. The notes are represented with integers, with central C on the piano keyboard denoted as 0, everything to the right of it as increasing integers, and everything to the left as decreasing integers. This is called a pitch-space. Next, we group the pitches into equivalence classes, by identifying the note X in all octaves to be the *same*. (So that all C's are 0, all D's are 1, and so on). Thus, we are working modulo 12. This is a "collapsed" pitch-space, because it is kind-of folded back onto itself, and the pitch-class space is isomorphic to the group $\mathcal{Z}_{12}$. We call it pitch-class space, or pc-space for short. The elements of the pc-space are denoted as 'pcs'.

6.1 Measuring Intervals

Given two ordered pitches a and b (not pitch-classes), we define the interval between them as one minus the other (b-a) (recall that we now represent pitches with integers). By ordered pitches (as opposed to unordered), we mean that one is played after the other

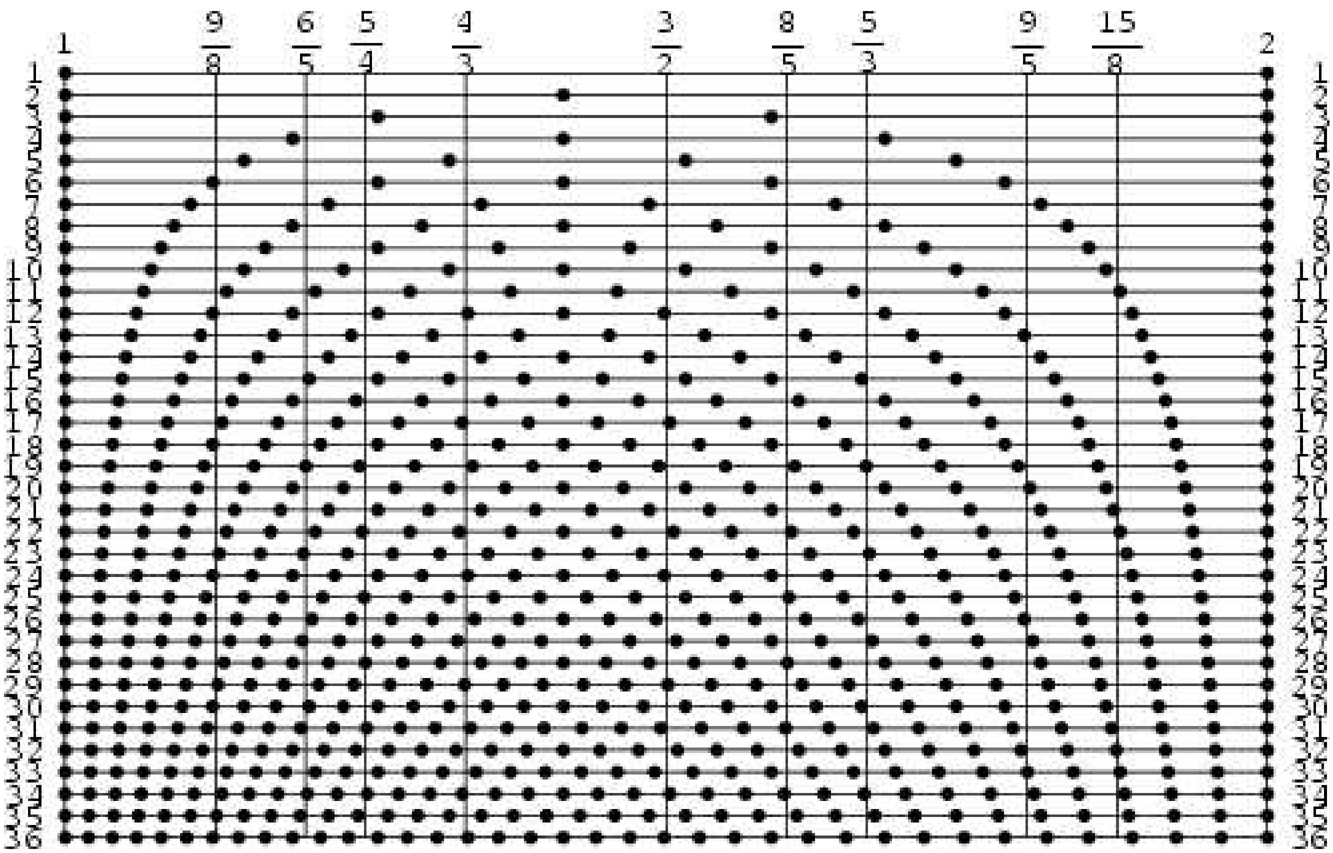

Figure 2: The y-axis lists possible choices of M, the number of subdivisions of the octave. The vertical lines cross at frequencies equal to rational numbers (just intonation) with small integer numerators and denominators. Notice that $M = 12$, $M = 19$ and $M = 31$ are good choices for M, because the equal tempered scale in each case matches many notes in the just intonation. This graphic is taken from [22].

in time[16]. Thus, the interval sign records the direction of change of pitch. However, if we are given two *unordered* pitches a and b, we must subtract in both directions, so that the "general" interval between the two is $+-(b-a)$.

A slight problem with the unordered scheme occurs when we work in pitch-class space. If a and b are now pcs (and not pitches), then $+-(b-a)$ is ambiguous. To see this, consider $a=3, b=5$. We have, $-2, +2$, but $-2=10$. So which one should we take as the value ? 10 or 2? To solve this problem, we introduce notation icX where ic stands for interval class. We always take the smallest positive value of the two: in above example, we will take 2 vs. 10 and denote it by writing $ic\{5,3\}=ic2$, where $ic\{a,b\}$ denotes the interval between pcs a and b. Clearly, there are only seven ics, $ic0$ through $ic6$.

6.2 Interval content of a pc-set

Now that we know how to compare two pcs, we may consider the intervals between every pair of notes in a set of pcs. Given such pc-set $A=\{a_0,a_1,\ldots,a_{n-1}\}$, the interval class *matrix* $T=[T_{ij}]$ of A is defined to have elements $T_{ij}=ic\{a_i,a_j\}$. In other words, this matrix records all intervals that occur in pc-set A. Note that this matrix is symmetric. Two pc-sets are *interval equivalent* if they have the same interval-class matrix.

The interval class vector of pc-set A denoted by $ICV(A)$ is a 7-tuple, $(l_0,l_1,l_2,\ldots,l_6)$, with l_X equal the numbers of times an interval of size icX occurs in the interval class matrix below the diagonal (and including the diagonal). Clearly, there may exist pc-sets that have different interval class matrices, but the same ICV vector (because information is lost during the transition from the interval-class matrix to the interval-class-vector). In such cases, we say that the two pc-sets are Z-related.

6.3 Twelve Tone Transformations

We now define transformations on pc-sets. The transposition T_n of a pc-set $A=\{a_0,a_1,\ldots,a_{n-1}\}$ is a new pc-set $B=\{b_0,b_1,\ldots,b_{n-1}\}$ with $B_i=A_i+n$.

The multiplication transformation M_n is defined similarly, but with $B_i=A_i\cdot n$. Note that to create a one-to-one mapping by multiplying numbers in $\mathcal{Z}_{12}$[17] we need n to be relatively prime to the modulus 12. Thus n can be $1,5,7$ or 11. Now, if we admit negative values of n, then $-5=7$, and $-1=11$, so that, essentially, M_{-1} and M_5 suffices. Note that T_n and M_m transformations can be cascaded. Clearly $M_mM_n(A)=M_{mn}(A)$. We also define operation of inversion I by $I=M_{-1}$, and with just M, the operation M_5. In can be shown that any combination of T_n's, and I's and M's can be written in one of the following forms: T_n, T_nI, T_nM, or T_MI.

Thus, we have operations M, T_n and I. These operators are called Twelve Tone Operators (TTOs) and they collectively form a group with T_0 being the identity operation. As well, T_n and one or none of I or M generate a subgroup.

Of the three operations, T_n and I do not change the interval class content of their operand. Let G be the group generated by those two operators. If two pc-sets are related by a transformation in this group, then they have the same equivalence class matrix, hence the same interval class vector.

We define an *equivalence relation* on pc-sets, by placing two pc-sets in the same equivalence class if they are related by a TTO in the group G. We denote each such pc-set class with $SC(i-j)[P]$ where i is the number of elements in the pc-set, j is the index of the SC within the same value of i (as listed in Allen Forte's SC table, see [23]), and P is a representative pc-set of the SC, called the prime form. For example, there are six different pc-sets classes consisting of two notes: $SC(2-1)[01]$, $SC(2-2)[02]$, $SC(2-3)[03]$, $SC(2-4)[04]$, $SC(2-5)[05]$, $SC(2-6)[06]$[18].

Another measure of a pc-set that is preserved by the TTOs in G is its invariance vector $IV(A)$. This vector is an 8-tuple, where the first four values show for how many different values of n the pc-set is invariant under transformations T_n, T_nI, T_nM, and T_nMI, respectively. The last four entries are similar. They ask, for how many different values of n does the pc-set map, under the above transformations, into its complement[19].

[16] This is not very different from the earlier definition of interval in the Greek sense, for we are implicitly taking the base 2 logarithm of the ratio and multiplying the result by M: $Mlog(2^{b/M}/2^{b/M})=b-a$.

[17] Equivalent notation, since we are working modulo 12.

[18] Allen Forte, an atonal music theorist, has categorized all SC's[23].

[19] The complement is taken with respect to $\mathcal{Z}_{12}$, i.e. the complement of {1,3,7} is {0,2,4,5,6,8,9,10,11}.

6.4 Prime Form Algorithm

Given a pc-set, we want to determine to which SC it belongs. The following algorithm transforms a pc-set to its prime form (and it is the prime-form that's listed in the SC table, and hence uniquely identifies its SC).

1 START: select pc-set $P = \{p_0, p_1, \ldots, p_{n_1}\}$.

2 $n := |P|$.

3 Arrange P and $I(P)$[20] in ascending order of pc number.

4 $q := 1$.

5 Produce the set S of pc-cycles[21] comprised of all rotations of P and $I(P)$ (S will have $2 \cdot |P|$ members).

6 Find the subset S' of all pc-cycles C in S where $i < C_0, C_{n-q} >$[22] is *minimal*.

7 $S := S'$ (delete all members of S not in S').

8 If S has only one member, go to 12.

9 If $q = n$, discard all (T_n-related) members of S except one, and go to 12.

10 $q := q + 1$.

11 Go to 6.

12 Transpose the (sole) member of S so it begins on 0.

13 END. the resulting pc-cycle is called the prime form of P.

6.5 Chains of SCs

Here we will create a sequence of pc-sets that are in different SCs, but with a special property. Take the union of any two adjacent pc-sets in this sequence, and it will always remain in the same SC!.

First we must introduce a concept of a partition of a pc-set. A two-partition of a pc-set is any way of dividing it into two non-intersecting parts. The two parts can be considered complements with respect to the pc-set. Thus, if $P = \{01347\}$, the following are all partitions, $\{01|347\}$, $\{07|234\}$ and so on. The general notation is $P = P1|P2$.

The chain of SCs therefore will look like this:

$$P1|P2|P3|\ldots$$

Thus, in a chain of SCs of our interest, we require $P_i|P_{i+1}$ to be in the same SC as $P_{i+1}|P_{i+2}$ for all i. (Note that we are chaining SC's with k elements, but every adjacent pair of k-element SCs are in the same $2k$-element SC. In the example below we have $k = 3$).

We also define operation of partition reversal $R(P1|P2) = P2|P1$.

Execution of this algorithm requires preparation of a list of *all partitions* of the prime-form of the SC that we would like to have as the main theme of the chain.

1 START: select initial two-partition from list. Call it the *current chain partition* (CCP).

2 The chain begins with $CCP_1|CCP_2$.

3 Let $BEG := CCP_1$.

4 Select another two-partition M from the list, where M_1 and M_2 is equivalent under a TTO F to CCP_2. (F is T_n or T_nI).

5 If $F(M_2) = CCP_2$ the let $NEW := R(M)$; else let $NEW := M$.

6 Transform NEW under F. $F(NEW)$ is called the *new chain partition* (NCP).

7 The chain continued: $(\ldots)CCP_1|CCP_2|NCP_1|NCP_2$

8 Let $CCP := NCP$

9 If CCP_2 is not equal to BEG, go to step 4; else stop—the chain is complete.

Notice that the stop condition is to eventually produce a loop, and come back to the beginning. This kind of "round" produces a sense of completeness.

[20] Recall that inversion $I = M_{-1}$

[21] A pc-cycle (pccyc for short) is an ordered list of pcs, that loops back on itself. Unlike conventional cycles in group theory, the starting element in a pc-cycle matters, so that (a, b, c) and (c, a, b) are different pccycs, although they are related by *rotation*.

[22] $i < a, b >= b - a$, that is difference between two ordered pitches (in time)

[23] Note that here A and B stand for numbers 10 and 11 respectively, and not for the notes 'la' and 'ti' as in conventional music theory.

If we run the above algorithm on $SC(6-42)[012369]$ with initial partition $CCP := 012|369$. The resulting chain is[23],

$$012|369|0AB|147|89A|B25|678|903|$$

$$456|7A1|234|58B|012|369$$

6.6 More...

This theory continues into the study of the group structure of TTOs, more in depth study of intervals, transformations of arrays, and further classification of SCs into K and Kh complexes. See [23].

7 Conclusion

Like mathematics, music is constantly evolving. And as history shows, music and mathematics have always been companions. Perhaps the comment by the mathematician J. Sylvester describes this relationship best,

> Mathematics is the Music of Reason.

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